Random fields, cosmology, and all that

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Motivation

A toy model

The convergence random field

Spin random fields

Back to the matter distribution

For further reading

Figure: What is that?

http://zebu.oregon.edu/hudf/
The components of the Universe (68% confidence limits)

1. Visible matter: $\Omega_b = 0.0486 \pm 0.001$
2. Dark matter: $\Omega_c = 0.2589 \pm 0.0057$
3. Radiation: $\Omega_r \approx 10^{-4}$
4. Dark energy: $\Omega_\Lambda = 0.6911 \pm 0.0062$

We would like to build the 3D map of matter distribution in our neighbourhood. We cannot see dark matter, but it deflects light of distant cosmic objects, and we can observe the corresponding effects.
## Light distortions in several languages: convergence

<table>
<thead>
<tr>
<th>telescope aberration</th>
<th>gravitational aberration</th>
<th>image</th>
<th>Zernike polynomial</th>
<th>Zernike graph</th>
</tr>
</thead>
<tbody>
<tr>
<td>Piston</td>
<td>Unlensed</td>
<td><img src="image" alt="image" /></td>
<td>(Z_0^0(\rho, \theta) = 1)</td>
<td><img src="graph" alt="graph" /></td>
</tr>
<tr>
<td>Tip</td>
<td>—</td>
<td>—</td>
<td>(Z_1^1(\rho, \theta) = 2\rho \cos(\theta))</td>
<td><img src="graph" alt="graph" /></td>
</tr>
<tr>
<td>Tilt</td>
<td>—</td>
<td>—</td>
<td>(Z_{-1}^1(\rho, \theta) = 2\rho \sin(\theta))</td>
<td><img src="graph" alt="graph" /></td>
</tr>
<tr>
<td>Defocus</td>
<td>Convergence, (\kappa)</td>
<td><img src="image" alt="image" /></td>
<td>(Z_2^0(\rho, \theta) = \sqrt{3}(2\rho^2 - 1))</td>
<td><img src="graph" alt="graph" /></td>
</tr>
</tbody>
</table>

**Table:** Light distortions I
**Light distortions in several languages: shear**

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<tr>
<td>Vertical Astigmatism</td>
<td>Shear, Re $\gamma$</td>
<td><img src="image" alt="Image" /></td>
<td>$Z_2^2(\rho, \theta) = \sqrt{6}\rho^2 \cos(2\theta)$</td>
<td><img src="image" alt="Graph" /></td>
</tr>
<tr>
<td>Oblique Astigmatism</td>
<td>Shear, Im $\gamma$</td>
<td><img src="image" alt="Image" /></td>
<td>$Z_2^{-2}(\rho, \theta) = \sqrt{6}\rho^2 \sin(2\theta)$</td>
<td><img src="image" alt="Graph" /></td>
</tr>
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**Table:** Light distortions II

Note: the distortion does not change under rotation by $\pi$. 
### Light distortions in several languages: $\mathcal{F}$-flexion

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<th>Zernike graph</th>
</tr>
</thead>
<tbody>
<tr>
<td>Horizontal Coma</td>
<td>$\mathcal{F}$-flexion, Re $\mathcal{F}$</td>
<td><img src="image1.png" alt="image" /></td>
<td>$Z_3^1(\rho, \theta) = \sqrt{8}(3\rho^3 - 2\rho) \cos(\theta)$</td>
<td><img src="graph1.png" alt="graph" /></td>
</tr>
<tr>
<td>Vertical Coma</td>
<td>$\mathcal{F}$-flexion, Im $\mathcal{F}$</td>
<td><img src="image2.png" alt="image" /></td>
<td>$Z_3^{-1}(\rho, \theta) = \sqrt{8}(3\rho^3 - 2\rho) \sin(\theta)$</td>
<td><img src="graph2.png" alt="graph" /></td>
</tr>
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**Table:** Light distortions III
**Light distortions in several languages: $\mathcal{G}$-flexion**

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<tbody>
<tr>
<td>Oblique</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Trefoil</td>
<td>$\mathcal{G}$-flexion, Re $\mathcal{G}$</td>
<td><img src="image1.png" alt="image" /></td>
<td>$Z_3^3(\rho, \theta) = \sqrt{8}\rho^3 \cos(3\theta)$</td>
<td><img src="graph1.png" alt="graph" /></td>
</tr>
<tr>
<td>Vertical</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Trefoil</td>
<td>$\mathcal{G}$-flexion, Im $\mathcal{G}$</td>
<td><img src="image2.png" alt="image" /></td>
<td>$Z_3^{-3}(\rho, \theta) = \sqrt{8}\rho^3 \sin(3\theta)$</td>
<td><img src="graph2.png" alt="graph" /></td>
</tr>
</tbody>
</table>

**Table:** Light distortions IV

Note: the distortion does not change under rotation by $2\pi/3$. 
Charting the Universe

In cosmology, the Universe is identified with a four-dimensional manifold, say $M$. It means that for each point $p \in M$, there is a chart: a one-to-one correspondence between a neighbourhood of $p$ and an open set in $\mathbb{R}^4$. The set of all charts, called the atlas, must satisfy some special conditions.

Consider an observer who perceives the Universe as isotropic. Consider a special chart $(t, r, n)$ that maps the observable part of the Universe onto a cylinder in $\mathbb{R}^4$. The observer’s location is mapped to the point $(t_0, 0) \in \mathbb{R}^4$, where $t_0$ is the elapsed time since the Big Bang according to a clock of the observer. A distant galaxy that has emitted light detected by the observer at time $t_e$ and has angular coordinates $n \in S^2$, is mapped to the point $(t_e, r, n)$, where $r$ is the distance to the galaxy now.

In the above chart, we can observe a single realisation of the isotropic random fields of convergence $\kappa(r, n)$, $\mathcal{F}$-flexion $\mathcal{F}(r, n)$, shear $\gamma(r, n)$, and $\mathcal{G}$-flexion $\mathcal{G}(r, n)$. (It is customary to omit the first coordinate, $t_e$, because it is completely determined by $r$.)
How to calculate $r$ and $t_e$?

We compare the observed wavelength $\lambda_{\text{obs}}$ of a distant source with the emitted wavelength $\lambda_{\text{em}}$ known from laboratory experiments. Define the redshift $z$ by

$$z = \frac{\lambda_{\text{obs}} - \lambda_{\text{em}}}{\lambda_{\text{em}}}$$

and let $a(t)$ be the scale factor of the Universe with $a(t_0) = 1$.

**Theorem**

The dynamics of $a(t)$ is defined by the Friedmann equation

$$\dot{a}(t) = H_0 \sqrt{(\Omega_b + \Omega_c) a^{-1}(t) + \Omega_r a^{-2}(t) + \Omega_\Lambda a^2(t)},$$

where $H_0$ is the Hubble constant. Moreover, we have

$$1 + z = \frac{1}{a(t_e)}, \quad r = c \int_{a(t_e)}^{1} \frac{da}{a \dot{a}}, \quad t_e = t_0 - \int_{a(t_e)}^{1} \frac{da}{\dot{a}}$$
The formulation of the problem

1. Construct a rigorous mathematical theory of the above fields.
2. Using their observed realisations, reconstruct the distribution of the matter in our neighbourhood (say, in a ball of radius 10 billion light years).

We will give a part of a solution to Problem 2 as long as the necessary notation will be introduced below.

Correct formulae for Problem 1 have been written by physicists since 1990s. Three variants of a rigorous theory were built (not independently!) by [Geller and Marinucci (2010)], [Malyarenko (2011)], and [Baldi and Rossi (2014)].
A toy model, or spherical horses in vacuum

Let $M = S^1$ be the centred unit circle embedded into the plane $\mathbb{R}^2$. Let $X(x)$ be a complex-valued random field on $S^1$, that is, there is a probability space $(\Omega, \mathcal{F}, P)$ and a function $X(x, \omega): S^1 \times \Omega \to \mathbb{C}$ such that for any fixed $x_0 \in S^1$ the function $X(x_0, \omega): \Omega \to \mathbb{C}$ is a random variable. Assume a little bit more: the function $X(x, \omega)$ is measurable as a function of two variables, and has a finite variance: $\mathbb{E}[|X(x)|^2] < \infty$ for all $x \in S^1$. In other words, $X(x) \in L^2(\Omega)$, where $L^2(\Omega)$ is the Hilbert space of all complex-valued random variables on $\Omega$ with inner product $(X, Y) = \mathbb{E}[X \overline{Y}]$.

By the result of [Marinucci and Peccati (2013)], the field $X(x)$ is mean-square continuous, that is, the map $X(x): S^1 \to L^2(\Omega)$ is continuous.

**Definition**

A random field $X(x)$ is called isotropic if its finite-dimensional distributions are invariant with respect to rotations.
Let $L^2(X)$ be the intersection of all closed subspaces of the space $L^2(\Omega)$ that contain the set $\{ X(x) : x \in S^1 \}$. The group of rotations $SO(2)$ acts on the vector $X(x)$ by

$$gX(x) = X(g^{-1}x).$$

This action can be extended by linearity and continuity to the action of $SO(2)$ in $L^2(X)$.

**Observation**

If the random field $X(x)$ is isotropic, then the above action is a unitary representation $U(g)$ of the group $SO(2)$, that is, a continuous homomorphism of $SO(2)$ to the group of unitary operators in $L^2(X)$ equipped with the strong operator topology: a sequence $U_n$ converges to $U$ if and only if for all $Y \in L^2(X)$ we have $\lim_{n \to \infty} \| U_n Y - UY \| = 0$. 
An inverse translation

Conversely, let $U$ be a unitary representation of the group $\text{SO}(2)$ in a Hilbert space $H$. Does it correspond to an isotropic random field on $S^1$?

The answer is no. However, we can describe all possible cases.

Let $L^2(S^1)$ be the Hilbert space of the square-integrable functions on $S^1$ with respect to the measure $d\varphi$. The operators

$$\tilde{U}(g)f(x) = f(g^{-1}x)$$

constitute a unitary representations of the group $\text{SO}(2)$ in the space $L^2(S^1)$ which is called the regular representation. We give an explicit construction of a unitary isomorphism $T$ between $L^2(X)$ and a closed subspace $H$ of $L^2(S^1)$ that intertwines the representations $U$ and the restriction $\tilde{U}\big|_H$ of the representation $\tilde{U}$ to its invariant subspace $H$ (that is, $\tilde{U}(g)Y \in H$ for all $g \in G$ and for all $Y \in H$).
The construction of the isomorphism $T$

Define

$$a_\ell = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} X(\varphi) e^{-i\ell \varphi} \, d\varphi, \quad \ell \in \mathbb{Z}. $$

Denote $C_\ell = \mathbb{E}[|a_\ell|^2]$ and $\mathbb{Z}_X = \{ \ell \in \mathbb{Z} : C_\ell > 0 \}$. Define $H$ as the Hilbert space with orthonormal basis $\{(2\pi)^{-1/2}e^{i\ell \varphi} : \ell \in \mathbb{Z}_X\}$ and a unitary operator $T: L^2(X) \to H$ by $Ta_\ell = \sqrt{C_\ell / (2\pi)}e^{i\ell \varphi}$. The diagram

$$
\begin{array}{c}
L^2(X) \xrightarrow{T} H \\
\downarrow U \quad \quad \downarrow \tilde{U}|_H \\
L^2(X) \xrightarrow{T} H
\end{array}
$$

is commutative: $U$ and $\tilde{U}|_H$ are equivalent representations.
The Fourier expansion of the random field $X$

The random variables $a_\ell$ are the Fourier coefficients of the random field $X(\varphi)$. We obtain the following representation:

$$X(\varphi) = \frac{1}{\sqrt{2\pi}} \sum_{\ell=-\infty}^{\infty} a_\ell e^{i\ell \varphi}, \quad \sum_{\ell=-\infty}^{\infty} C_\ell < \infty.$$  

It is easy to see that the random field $X(\varphi)$ is real-valued if and only if $a_{-\ell} = \overline{a_\ell}$. In particular, $a_0$ is real-valued. In this case we have

$$X(\varphi) = \frac{1}{\sqrt{2\pi}} a_0 + \sqrt{\frac{2}{\pi}} \sum_{\ell=1}^{\infty} \Re a_\ell \cos(\ell \varphi) - \sqrt{\frac{2}{\pi}} \sum_{\ell=1}^{\infty} \Im a_\ell \sin(\ell \varphi).$$
Adding the distance

Consider the case of a mean-square continuous random field $X(x)$ defined on the plane $\mathbb{R}^2$. Call it isotropic if its finite-dimensional distributions are invariant with respect to rotations. The spectral expansion of such a field was obtained by [Yadrenko (1963)]. His idea is as follows. The restriction of the random field $X(x)$ to the centred circle of radius $r$ is an isotropic field on this circle and therefore has the form

$$X(r, \varphi) = \frac{1}{\sqrt{2\pi}} \sum_{\ell=-\infty}^{\infty} a_\ell(r) e^{i\ell \varphi}.$$ 

The stochastic processes $a_\ell(r)$ are mean-square continuous with

$$E[a_\ell(r)] = \sqrt{2\pi} E[X(x)] \delta_{\ell 0}, \quad E[a_\ell(r_1) \overline{a_{\ell'}(r_2)}] = B_\ell(r_1, r_2) \delta_{\ell\ell'}$$

with

$$\sum_{\ell=-\infty}^{\infty} B_\ell(r, r) < \infty, \quad r > 0.$$
A more realistic model

The suitable basis of the space $L^2(S^1)$ consists of the functions proportional to the matrix entries $e^{i\ell \varphi}$ of the irreducible unitary representations of the corresponding symmetry group, $SO(2)$. Only in this basis, the Fourier coefficients $a_{\ell}$ of an isotropic random field are uncorrelated.

Similarly, the suitable basis of the space $L^2(S^2)$ consists of the spherical harmonics $Y_{\ell m}(\mathbf{n})$ proportional to the matrix entries of the irreducible unitary representations $U_{\ell}$ of the corresponding symmetry group, $SO(3)$. Only in this basis, the Fourier coefficients $a_{\ell m}$ of an isotropic random field are uncorrelated:

$$X(\mathbf{n}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}(\mathbf{n}), \quad \mathbb{E}[a_{\ell m} a_{\ell' m'}] = \delta_{\ell \ell'} \delta_{mm'} C_{\ell}$$

with $\sum_{\ell=0}^{\infty} (2\ell + 1) C_{\ell} < \infty$. 
The convergence random field

This field is not changing when the instrument rotates. Similarly, it has the form

$$\kappa(r, n) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m}(r) Y_{\ell m}(n). \quad (1)$$

The stochastic processes $a_{\ell m}(r)$ are given by

$$a_{\ell m}(r) = \int_{S^2} \kappa(r, n) \overline{Y_{\ell m}(n)} \, dn$$

with

$$E[a_{\ell m}(r)] = E[\kappa(r, n)] \delta_{\ell 0}, \quad E[a_{\ell m}(r_1) a_{\ell' m'}(r_2)] = B_{\ell}(r_1, r_2) \delta_{\ell \ell'} \delta_{mm'}$$

and

$$\sum_{\ell=0}^{\infty} (2\ell + 1) B_{\ell}(r, r) < \infty, \quad r > 0.$$
A construction of a fiber bundle

Realise the group $K = \text{SO}(2)$ as the subgroup of rotations $g_\varphi$ by the angle $\varphi$ about z-axis in the group $G = \text{SO}(3)$. The map $U(\varphi) = e^{is\varphi}$, $s \in \mathbb{Z}$, is a unitary representation of the group $K$ in the space $\mathbb{C}^1$. Introduce the following equivalence relation in the Cartesian product $G \times \mathbb{C}^1$: $(g_1, z_1) \sim (g_2, z_2)$ if and only if there is a $g_\varphi \in K$ with $g_2 = g_1 g_\varphi$ and $z_2 = e^{-is\varphi} z_1$. Denote the set of equivalence classes by $G \times_K \mathbb{C}^1$, and let $[g, z] \in G \times_K \mathbb{C}^1$ be the equivalence class of the point $(g, z) \in G \times \mathbb{C}^1$.

The set of left cosets $gK = \{ gk : k \in K \}$ is the sphere $G/K = S^2$. Define the projection map $\pi: G \times_K \mathbb{C}^1 \rightarrow S^2$ by $\pi([g, z]) = gK$. The triple $(G \times_K \mathbb{C}^1, \pi, S^2)$ is a fiber bundle.
A simple exercise

Exercise

Prove that the triple \((G \times_K \mathbb{C}^1, \pi, S^2)\) indeed satisfies the definition of the fibre bundle:

1. \(G \times_K \mathbb{C}^1\) is a smooth manifold;
2. \(\pi(G \times_K \mathbb{C}^1) = S^2\);
3. for each \(n \in S^2\), the fiber \(\pi^{-1}(n)\) is a copy of \(\mathbb{C}^1\);
4. there is a neighbourhood \(U\) of \(n\) in \(S^2\) and a diffeomorphism \(\phi: \pi^{-1}(U) \to U \times \mathbb{C}^1\) with \(\phi(\pi^{-1}(m)) = (m, \mathbb{C}^1)\) for all \(m \in U\).
Exercise

Prove that under the action of $G$ given by $g_1(gK, z) = (g_1gK, z)$ for $g_1 \in G$, the fibre bundle $(G \times_K \mathbb{C}^1, \pi, S^2)$ is homogeneous, that is

1. the action of $G$ on $G \times_K \mathbb{C}^1$ preserves fibers: for each $g \in G$ and $n \in S^2$ there is $m \in S^2$ with $g\pi^{-1}(n) \subseteq \pi^{-1}(m)$;

2. the descended action $gn = m$ is transitive;

3. each $g \in G$ maps $\pi^{-1}(n)$ to $\pi^{-1}(gn)$ linearly for all $n \in S^2$. 
**From homogeneous bundles to representations**

**Definition**

Two fibre bundles \((V, \pi, M)\) and \((V', \pi', M)\) are **equivalent** if there is a diffeomorphism \(\phi: V \rightarrow V'\) such that for each \(x \in M\) \(\phi\) maps \(\pi^{-1}(x)\) to \(\pi'^{-1}(x)\) linearly and \(\pi' \phi = \pi\).

**Exercise**

Let \((V, \pi, M)\) be a homogeneous vector bundle for \(G\) on \(M = G/K\) with the fiber \(H\), and let \(e\) be the identity element of \(G\). Prove the following:

1. the fiber \(\pi^{-1}(eK)\) carries the representation \(U(k)h = kh\) of the group \(K\), \(h \in H\);
2. If \((V', \pi', M)\) is another fibre bundle and \(\phi\) is an equivalence diffeomorphism between \(V\) and \(V'\), then the restriction of \(\phi\) to \(V_{eK}\) intertwines the representations \(U\) and \(U'\).
Sections

A measurable section of the bundle \((V, \pi, M)\) is a measurable map \(f: M \to V\) such that \(\pi f: M \to M\) is the identity map. A measurable section \(f\) of the bundle \((G \times K \mathbb{C}^1, \pi, S^2)\) is called square-integrable if

\[
\int_{S^2} |f(n)|^2 \, dn < \infty.
\]

The set \(L^2(S^2, G \times K \mathbb{C}^1)\) of all square-integrable sections becomes a Hilbert space under the inner product

\[
(f_1, f_2) = \int_{S^2} f_1(n)\overline{f_2(n)} \, dn.
\]
Induced representations

Let $f$ be a square-integrable section of the bundle $(G \times_K \mathbb{C}^1, \pi, S^2)$ defined by

$$f(g'K) = [g', f_{g'}].$$

The induced representation $(\text{Ind}_K^G(U))(g)$ acts on the above section by

$$(\text{Ind}_K^G(U)(g)f)(g'K) = [gg', f_{g'}], \quad g \in G.$$
The structure of the induced representation

Let $W$ be a representation of $G$ in a finite-dimensional space $L$. How many copies of $W$ are containing in $\text{Ind}_K^G(U)$?

**Theorem (Frobenius reciprocity)**

The number of copies of $W$ containing in $\text{Ind}_K^G(U)$ is equal to the number of copies of $U$ in the restriction $W|_K$.

**Example**

The representation $U(\varphi) = e^{is\varphi}$ of the group $K = \text{SO}(2)$ is containing once is each representation $U^{\ell}$ of the group $G = \text{SO}(3)$ if $\ell \geq |s|$, and is not contained otherwise. It follows that the induced representation $\text{Ind}_{\text{SO}(2)}^{\text{SO}(3)}(U)$ is equivalent to the direct sum of the representations $U^{\ell}$, $\ell \geq |s|$.
The suitable basis of the space $L^2(S^2, G \times_K C^1)$ consists of the spin spherical harmonics $s Y_{\ell m}(n)$ proportional to the matrix entries of the components $U^\ell$, $\ell \geq |s|$. Only in this basis, the Fourier coefficients $s a_{\ell m}$ of a spin $s$ isotropic random field are uncorrelated:

$$X(n) = \sum_{\ell = |s|}^{\infty} \sum_{m = -\ell}^{\ell} s a_{\ell m} s Y_{\ell m}(n), \quad E[s a_{\ell m} \overline{s a_{\ell' m'}}] = \delta_{\ell \ell'} \delta_{mm'} s C_{\ell}$$

with $\sum_{\ell = |s|}^{\infty} (2\ell + 1)_s C_{\ell} < \infty$. 
Shear and flexions

In particular, the $\mathcal{F}$-flexion is a spin 1 isotropic random field:

$$\mathcal{F}(r, n) = \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m}(r) Y_{\ell m}(n).$$

(2)

The shear is a spin 2 isotropic random field:

$$\gamma(r, n) = \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} 2a_{\ell m}(r) Y_{\ell m}(n).$$

(3)

Finally, the $\mathcal{G}$-flexion is a spin 3 isotropic random field:

$$\mathcal{G}(r, n) = \sum_{\ell=3}^{\infty} \sum_{m=-\ell}^{\ell} 3a_{\ell m}(r) Y_{\ell m}(n).$$

(4)
The operator $\delta$

Let $C^\infty_\mathcal{S}$ be the space of infinitely differentiable sections of the fiber bundle corresponding to the representation $U(\varphi) = e^{is\varphi}$ of the group $K = SO(2)$. The operator (in fact, the set of operators) $\delta$ acts on a section $f(\varphi, \theta)$ by

$$\delta f(\varphi, \theta) = s \cot \theta - \frac{\partial f(\varphi, \theta)}{\partial \theta} - \frac{i}{\sin \theta} \frac{\partial f(\varphi, \theta)}{\partial \varphi}.$$ 

The operator $\delta^*$ acts by

$$\delta^* f(\varphi, \theta) = s \cot \theta - \frac{\partial f(\varphi, \theta)}{\partial \theta} + \frac{i}{\sin \theta} \frac{\partial f(\varphi, \theta)}{\partial \varphi}.$$
Spin raising and spin lowering

Theorem

The operator $\bar{\sigma}$ raises the spin:

$$\bar{\sigma}_s Y_{\ell m}(n) = \begin{cases} 0, & \text{if } \ell = s \geq 0, \\ \sqrt{(\ell - s)(\ell + s + 1)} Y_{s+1}(n), & \text{otherwise}. \end{cases}$$

The operator $\bar{\sigma}^*$ lowers the spin:

$$\bar{\sigma}^*_s Y_{\ell m}(n) = \begin{cases} 0, & \text{if } \ell = -s \geq 0, \\ -\sqrt{(\ell + s)(\ell - s + 1)} Y_{s-1}(n), & \text{otherwise}. \end{cases}$$
The density contrast and the lensing potential

**Definition**

The density contrast is \( \delta(r, n) = \frac{\rho(r, n) - \rho_0}{\rho_0} \), where \( \rho(r, n) \) is the matter density and \( \rho_0 \) is the mean matter density. The lensing potential is

\[
\phi(r, n) = \frac{2}{c^2} \int_0^r \frac{r' - r'}{rr'} \Phi(r', n) \, dr',
\]

where the Newton potential \( \Phi(r', n) \) is the solution to Poisson’s equation

\[
\nabla^2 \Phi(r, n) = \frac{3(\Omega_b + \Omega_c)H_0^2}{2a(r)} \delta(r, n).
\]

Observe that the potentials are not homogeneous random fields as they are viewed on the past light cone.
A connection with observables

Theorem

We have

\[
\kappa(r, n) = \frac{1}{4}(\bar{\phi} \bar{\phi}^* + \bar{\phi}^* \bar{\phi}) \phi(r, n),
\]

\[
\mathcal{F}(r, n) = -\frac{1}{6}(\bar{\phi}^* \bar{\phi} \bar{\phi} + \bar{\phi} \bar{\phi}^* \bar{\phi} + \bar{\phi} \bar{\phi} \bar{\phi}^*) \phi(r, n),
\]

\[
\gamma(r, n) = \frac{1}{2} \bar{\phi}^2 \phi(r, n),
\]

\[
\mathcal{G}(r, n) = -\frac{1}{2} \bar{\phi}^3 \phi(r, n).
\]

Using Equations (1)–(4), we can recover any field by observing another field as a part of mass-maps technique, see [Wallis et al (2017)].
The Fourier–Bessel transform

Assume that the trajectories of the stochastic processes $s a_{\ell m}(r)$ are almost surely square-integrable. Define the Fourier–Bessel transform of the process $s a_{\ell m}(r)$ by

$$s \hat{a}_{\ell m}(k) = \sqrt{2/\pi} \int_0^\infty s a_{\ell m}(r) j_\ell(kr)r^2 \, dr,$$

where $j_\ell(kr)$ are the spherical Bessel functions, and $k$ is the radial wavenumber. Equations (1)–(4) take the form

$$s X(r, n) = \sum_{\ell = |s|}^{\infty} \sum_{m = -\ell}^{\ell} \sqrt{2/\pi} \int_0^\infty s \hat{a}_{\ell m}(k) j_\ell(kr) k^2 \, dk_s \, Y_{\ell m}(n),$$

which gives the representation of the generic field $s X(r, n)$ in the harmonic space: the union of countably many copies of the radial wavenumber domain $[0, \infty)$ indexed by the pairs $(\ell, m)$: $\ell \geq s$, $-\ell \leq m \leq \ell$. 

For further reading...

1.32
Why the Fourier–Bessel transform is useful and why not?

The basis functions \( j_\ell(kr) Y_{\ell m}(n) \) of this transform are eigenfunctions of the Laplacian operator \( \nabla^2 \) with eigenvalues \(-k^2\):

\[
\nabla^2 j_\ell(kr) Y_{\ell m}(n) = -k^2 j_\ell(kr) Y_{\ell m}(n).
\]

Therefore, in the harmonic space Poisson’s equation takes the simple form:

\[
\Phi_{\ell m}(k, r) = \frac{3(\Omega_b + \Omega_c) H_0^2}{2k^2 a(r)} \delta_{\ell m}(k, r),
\]

where \( \Phi_{\ell m}(k, r) \) (resp. \( \delta_{\ell m}(k, r) \)) is the Fourier–Bessel transform of the homogeneous field \( \Phi'(r, n) \) (resp. \( \delta'(r, n) \)) existing everywhere at the time \( t \) when the observed light has been emitted by a galaxy at distance \( r \). On the other hand, the Fourier–Bessel transform is difficult to evaluate because it does not admit a sampling theorem.
Instead of integrals over $[0, \infty)$, it is enough to consider integrals over $[0, R_{\text{max}}]$, where $R_{\text{max}}$ is defined by the sensitivity of the hardware.

Instead of using the basis $\{ j_\ell (kr)_{,s} Y_{\ell m}(n) \}$, one can combine the angular basis of spin spherical harmonics with a radial basis on a finite interval that admits a sampling theorem. It is a subject of ongoing research in cooperation with Professor Nikolai Leonenko from Cardiff University, UK.
Motivation

A toy model

The convergence random field

Spin random fields

Back to the matter distribution

For further reading


Random fields, cosmology, and all that

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http://mymorningcoast.com/2015/07/give-thanks/